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The Difference Between IRR and NPV in Capital Investment Appraisals

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Abstract

In the fields of architecture, engineering and construction, a clear concept of the appraisal methods of capital investment alternatives is very important. The NPV ranking of mutually exclusive alternatives is a correct approach if the MARR (minimum attractive rate of return) is based. The IRR ranking is an incorrect approach. If the IRR methodis ever used to rank alternatives, the Incremental IRR Analysis must be used, and its result will be the same as the NPV ranking. In using the Incremental IRR Analysis, one may face the problem of multiple IRRs, but this is not reallya hindrance to the Incremental IRR Analysis. The multiple IRR problem alleged in some articles is probably due to a misunderstanding of the multiple IRR theory. The author attempts to explain it in this article. The conclusion states that all other methods such as Modified IRR, Marginal Growth Rate, Incremental IRR Analysis, etc., are supplementary to the NPV method. The NPV is an economic indicator and always correct in evaluating the economic value of an investment and in ranking mutually exclusive alternatives based on the MARR. The IRR is a financial indicator and incorrect to be used for ranking mutually exclusive alternatives, but is to be used for finding the best financial strategy to achieve optimal gain for a single investment alternative.

Keywords

Engineering Management; Engineering Economics; Internal Rate of Return; Net Present Value; IRR; NPV

1. Introduction

An article entitled "The variable financial indicator IRR and the constant economic indicator NPV" (Tang and Tang 2003) was published in *The Engineering Economist*. For the past 15 years (from 2004 to 2018), there has been a number of citations of the said article, both in *The Engineering Economist* and in other international journals. The concept that IRR is a financial indicator and NPV an economic indicator was firstly proposed in 1991 in a book entitled *Economic Feasibility of Projects*, the 1st edition of which was published by McGraw-Hill (Tang 1991, Chapter 5). This current article is to respond to a number of important questions found in a number of important articles in the past 15 years (2004 - 2018), many of which cited the article written by Tang and Tang (2003). The responses in this current article, written in 2019/20, are mainly made to Hajdasinski (2004), Magni (2010), Robison *et al*. (2015), and Lefley (2018), besides others.

In the fields of architecture, engineering and construction, a clear concept of financial and economic appraisals of capital investment alternatives is very important. The word "financial" usually relates to private investors' perspective and "economic" to society's (or community's) perspective. This article is written with the following arrangement. It starts with the discussion of whether or not ranking mutually exclusive investment alternatives by ranking their IRRs is justified. It is a misunderstanding made by Hajdasinski (2004) that the article (Tang and Tang 2003) implied that ranking alternatives by ranking their IRRs is justified. The current author says it is a misunderstanding because Tang and Tang (2003) never justified this, and on the contrary, said IRR ranking could lead to wrong conclusions. This point is to be clarified in this article with the use of an example, in which both the direct IRR method and the Incremental IRR Analysis are used to rank five alternatives. Then, the Incremental IRR Analysis is applied again to rank three other cash flow streams, which were taken by Hajdasinski (2004) from (Tang and Tang 2003) for discussion in the former's 2004 article. The result of the ranking using the Incremental IRR Analysis shows that these three cash flow streams are equally good (equal ranking). This result is the same as that of the NPV method – same economic values (or economic returns) of the three cash flow streams. Such an observation is important because the three cash flow streams are derived from a single (the same) investment cash flow stream but with three different financial arrangements. The three cash flow streams so derived have different IRRs but the same NPVs.

The phenomenon of having identical NPVs but different IRRs can be interpreted as: the NPV is an economic indicator and the IRR a financial indicator.

Following that, the discussion is devoted to resolving the problem when multiple IRRs occur during the performance of the Incremental IRR Analysis. This question was raised by Hajdasinski (2004) in his responses to Tang and Tang (2003). In fact, Hazen (2003) had mentioned that each IRR of the multiple IRRs is meaningful, explainable and non-contradictory, and is consistent with the NPV evaluation of the investment cash flow stream. Hajdasinski's own example in his own article (Hajdasinski 2004) is used in this current article to show how Hazen's method can satisfactorily tackle Hajdasinski's multiple IRR question. Moreover, the multiple IRR problem brought out in Magni (2010) is also discussed.

After that, Robison *et al*.'s (2015) article entitled "Consistent IRR and NPV rankings" is looked into. That article attempts to find a consistent IRR and NPV ranking, or more exactly, consistent MIRR (modified IRR) and MNPV (modified NPV) ranking (Robison *et al*. 2015; Lefley 2018). There are, however, two major drawbacks in their method when it is applied in capital investment appraisals; the details are to be discussed later in this current article.

In summary, the author's purpose of writing this current article is to respond to a number of important questions about the IRR and the NPV found in a number of important articles in the past 15 years (from 2004 to 2018; this paper was submitted in May 2020 and the conference has been, due to the covid-19 incident, postponed several times to May 2022). During responding to the questions, that the NPV is an economic indicator and the IRR a financial indicator is further explained and reaffirmed.

2. The Incremental Analysis

In Hajdasinski article (Hajdasinski 2004), there was a misunderstanding that Tang and Tang (2003) implied ranking investment alternatives by ranking their IRRs is justified. Having said that, the author agrees (also agreed by Hajdasinski) that the incremental IRR ranking approach is necessary to remedy this shortcoming (i.e., possible wrong ranking of alternatives by just using direct IRR ranking). The following example shows the different ranking results of using the direct IRR ranking and the incremental IRR ranking.

Example: Rank the five investment alternatives shown in Table 1 using IRR ranking and NPV ranking:

Table 1: NCFs (Net Cash Flows in million$) of five mutually exclusive alternatives, IRRs and NPVs

End of year	Alternative A	Alternative B	Alternative C	Alternative D	Alternative E
0	-50	-70	-110	-140	-180
1	5.2	7.0	9.0	13.5	17.0
2	5.2	7.0	9.0	13.5	17.0
: : :	: : :	: : :	: : :	: : :	: : :
30	5.2	7.0	9.0	13.5	17.0
Direct IRR	9.8% p.a.	9.3% p.a.	7.2% p.a.	8.9% p.a.	8.7% p.a.
IRR Ranking	1st	2nd	5th	3rd	4th
NPV	21.57	26.35	13.88	45.83	54.00
NPV Ranking	4th	3rd	5th	2nd	1st

(The discount rate or the minimum attractive rate of return i is taken as 6% p.a.)

From Table 1, it can be seen that the direct IRR ranking of the five alternatives has a different result from the NPV ranking. Only the NPV ranking is correct and the IRR ranking is not (see reasons in the paragraphs below). While Lefley (2018, page 48) correctly quotes from Tang & Tang (2003) and Hajdasinski (2004) "the two models are said to have intrinsic differences from each other, with the NPV being an economic indicator and the IRR a financial indicator of a capital investment", he however is not so correct to say "when considering mutually exclusive projects, on the basis of the NPV, the project with the highest NPV would be accepted, while with respect to the IRR, the project with the greatest IRR/yield would be accepted". Such a statement is true in the first half but not in the secondhalf. The NPV ranking is indeed a correct approach but the IRR ranking - the alternative with the greatest IRR/yield would be accepted - is not a correct approach. The same incorrect statement can be found on page 500 of Robison *et* *al*. (2015). It is said there "Alternatively, investments (assumed to be mutually exclusive – a judgment from the context on that page) can be ranked using the challengers' IRRs" (for the meaning of challenger please see below). This is exactly the IRR ranking approach that this current author does not agree with.

Now, this current author would like to show how the result of the Incremental IRR Analysis is different from that of the IRR ranking. The "Incremental IRR" is a term or name given to the IRR calculated from a series of Incremental NCFs (Incremental Net Cash Flows), where an Incremental NCF is equal to the NCF of the j^{th} alternative minus the NCF of the $(j\text{-}1)^{th}$ alternative. In the analysis, the alternative with the lowest initial capital cost is arranged as the first alternative and the one with the highest as the last. The following criteria are adopted in the analysis:

If Incremental IRR > i, then the j^{th} alternative is better than the $(j\text{-}1)^{th}$ alternative, and
if Incremental IRR = i, then the $(j\text{-}1)^{th}$ alternative and the j^{th} alternative are equally good, and
if Incremental IRR < i, then the $(j\text{-}1)^{th}$ alternative is better than the j^{th} alternative,
where i is the discount rate used or the minimum attractive rate of return.

As a remark, the above criteria are only true for a series of Incremental NCFs that has one real Incremental IRR only (i.e., no multiple Incremental IRRs). If a series of Incremental NCFs has multiple Incremental IRRs, the analysis can still be performed. In that case, another set of criteria will be used instead of the set shown above. This important point will be further elaborated in this article.

To start to perform the Incremental IRR Analysis, compare Alternatives A and B as shown in Table 2:

Table 2: Incremental NCFs in million$ of (Alternative B minus Alternative A)

End of year	Alternative A	Alternative B	$(\text{Incremental NCF})_{B\text{-}A}$
0	-50	-70	-20
1	5.2	7.0	1.8
2	5.2	7.0	1.8
: : :	: : :	: : :	: : :
30	5.2	7.0	1.8

Some people may like to call Alternative A a defender and Alternative B a challenger. The Incremental IRR calculated from the series of Incremental NCFs (-20; 1.8; 1.8; … ; 1.8) is 8.1% p.a. Since 8.1% p.a. > 6% p.a. (the minimum attractive rate of return) and according to the above-said criteria, Alternative B is better than Alternative A. So, Alternative A is out. Next, compare Alternatives B and C.

This time, Alternative B is a defender and Alternative C a challenger. (In fact, the author considers that to distinguish between a defender and a challenger in many cases (e.g. this example) is unnecessary). The Incremental NCFs in million$ of (Alternative C minus Alternative B) are represented by -40 at the End of Year 0 and 2.0 each year from the End of Year 1 to the End of Year 30 (refer to Table 1), forming a series of Incremental NCFs (-40; 2.0;2.0; … ; 2.0). The Incremental IRR calculated is 2.8% p.a. Since 2.8% p.a. < 6% p.a., Alternative B is better than Alternative C according to the above-said criteria. So, Alternative C is out. Next, proceed with comparing Alternative B with Alternative D.

In a similar manner, the calculation shows the Incremental IRR of the $(\text{Incremental NCFs})_{D\text{-}B}$ to be 8.5% p.a. Since 8.5% p.a. > 6% p.a., Alternative D is better than Alternative B. Alternative B is out.

Following this, repeat calculating the Incremental IRR of the $(\text{Incremental NCFs})_{E\text{-}D}$, which is 7.8% p.a. Since 7.8% p.a. > 6% p.a., Alternative E is better than Alternative D. Hence, Alternative E is the best.

From the above Incremental IRR Analysis, one can see that the result of the ranking of the five alternatives is the same as the NPV ranking in Table 1. It confirms what Hajdasinski (2004) said – a direct comparison of the IRRs is not a recommended approach and the incremental ranking approach is needed. This is also what the author agrees with. As a matter of fact, it has been said in Tang (1991, Chapter 5, page 79) and Tang (2003, Chapter 5, page 80) that "In conclusion, in the economic appraisal of projects, the best way for selecting alternatives is to use NPV method or the Incremental Analysis. The direct IRR method may lead to wrong decisions". Both places were written earlier than the article written by Hadjasinski (2004). So, it was definitely a misunderstanding made by Hadjasinski (2004) to say that Tang and Tang (2003) implied ranking investment alternatives by ranking their IRRs is justified. In fact, Tang and Tang (2003) did not justify this or imply to justify this.

It should be noted another phenomenon happens if i = 9% p.a. (instead of 6% p.a.) is used to calculate the NPVs of the five alternatives in Table 1. Table 3 below shows the results of using this new i (9% p.a.):

Table 3: NCFs (Net Cash Flows in million$) of five mutually exclusive alternatives, IRRs and NPVs

End of year	Alternative A	Alternative B	Alternative C	Alternative D	Alternative E
0	-50	-70	-110	-140	-180
1	5.2	7.0	9.0	13.5	17.0
2	5.2	7.0	9.0	13.5	17.0
: : :	: : :	: : :	: : :	: : :	: : :
30	5.2	7.0	9.0	13.5	17.0
Direct IRR	9.8% p.a.	9.3% p.a.	7.2% p.a.	8.9% p.a.	8.7% p.a.
IRR Ranking	1st	2nd	5th	3rd	4th
NPV	3.42	1.92	-17.54	-1.31	-5.35
NPV Ranking	1st	2nd	5th	3rd	4th

(The discount rate or the minimum attractive rate of return i is taken as 9% p.a.)

It can be observed that the IRR ranking is the same as the NPV ranking. This phenomenon is to be explained later. It is also important to observe that the Incremental IRR Analysis has a new ranking result similar to Table 3 after i is changed to 9% p.a. Readers can verify these by themselves using the method just described. The result of the NPV ranking is always consistent with that of the Incremental Analysis.

Another situation that the IRR ranking is always the same as the NPV ranking is when the initial capital costs of all the investment alternatives are equal. In this situation, the alternatives are no longer called mutually exclusive alternatives because they have the same datum for comparison – same capital outlays. Because they are not mutually exclusive and have a same basis (same initial capital costs) for comparison, the ranking of their direct IRRs is sufficient to be a reference to know their true ranking and this ranking must be consistent with the ranking of their NPVs. Readers may refer to Tang (1991, page 75) or Tang (2003, page 77) for the detailed discussion.

Table 4 below shows an example of such a situation:

Table 4: NCFs (Net Cash Flows in million$) of three alternatives with similar initial investments, IRRs and NPVs

End of year	Alternative A	Alternative B	Alternative C
0	-30	-30	-30
1	4	9	10
2	5	9	8
3	6	9	7
4	7	9	7
5	8		6
6	9		
Direct IRR	7.04% p.a.	7.71% p.a.	9.24% p.a.
IRR Ranking	3rd	2nd	1st
NPV at 6% p.a.	1.13	1.19	2.46
NPV Ranking	3rd	2nd	1st
NPV at 9% p.a.	-1.96	-0.84	0.17
NPV Ranking	3rd	2nd	1st

(The discount rates or the minimum attractive rates of return i are taken as 6% p.a. and 9% p.a.)
(6% p.a. and 9% p.a. are taken arbitrarily)

From the example illustrated in Table 4, readers can see that the IRR ranking is always consistent with the NPV ranking for such so called "homogeneous investment size" alternatives. This statement, however, is not applicable to alternatives with multiple IRRs. An example is given by Robison *et al*. (2015, pages 500-501). Two homogeneous investment size alternatives, (-4, 12, -9) and (-4, 12, -8) are investigated, and it is found that the IRR ranking is inconsistent with the NPV ranking. This is because the alternatives have multiple IRRs. This point will be further discussed later in this article.

3. Comparing the three Cash Flow Patterns of three different financial arrangements

Hajdasinski (2004, pages 186-187) continued to discuss the numerical example of Tang and Tang (2003, pages 71-73). (1) below is the CFP (cash flow pattern) of an all-equity case (no borrowing) investment:

(-10000; 5000; 5000; 5000) (1)

Then, (2), a PFA (project financing alternative) where the original 10,000 investment that is financed in full by the equity capital as shown in (1) is now created by leveraging (1) with a market borrowing of 6,000 at i = 10% per period, is as follows:

(6000; -2600; -2400; -2200) (2)

The above PFA (2) is derived as shown in Table 5:

Table 5: Calculation of project financing alternative (2), i.e., PFA (2)

End of period	(1) Amount borrowed	(2) Principal amortization	(3) Balance of principal unpaid	(4) Interest (10% per period)	(5) Total periodic payment (2)+(4)
0	6,000		6,000		
1		2,000	4,000	600	2,600
2		2,000	2,000	400	2,400
3		2,000	0	200	2,200

As a result of introducing PFA (2), CFP (1) becomes CFP (3) as shown in (3):

(-4000; 2400; 2600; 2800) (3)

If (1) is leveraged even further, that 1,000 equity capital and 9,000 borrowing are combined to form the initial investment capital of 10,000, the PFA this time is (9000; -3900; -3600; -3300) as calculated by a similar method as shown in Table 5, and CFP (1) becomes CFP (4):

(-1000; 1100; 1400; 1700) (4)

CFPs (1), (3) and (4) in fact represent the same investment but with different financial arrangements (i.e., different PFAs). The NPVs of these three CFPs are the same, equal to 2,434.26 each. This represents that the economic value (or economic return) of the investment is 2,434.26 so there is only one economic value (or economic return) for a single investment. (The author would like to highlight Lefley's good and prudent wording "The NPV of 279,xxx is said to be the economic return from the investment" in his article recently published (Lefley 2018, page 49)). The IRRs of CFPs (1), (3) and (4), however, are 23.4%, 41.2% and 113.1% per period respectively. This does not mean that economically (4) is better than (3) and (3) is better than (1), which is the result of the IRR ranking. This is apparently Hajdasinski's (2004) misunderstanding on what was conveyed by the article of Tang and Tang (2003). Therefore, a clarification is necessary on what was really conveyed by Tang and Tang's (2003) article. The clarification is shown below:

First, use the Incremental IRR Analysis to compare CFPs (1), (3) and (4).

Initially, compare CFPs (3) and (4), and the (Incremental NCFs)$_{3\text{-}4}$ are shown in (5):

(-3000; 1300; 1200; 1100) (5)

The Incremental IRR calculated from (5) is 10% per period. It is exactly equal to the borrowing rate i (or treated here as the MARR (minimum attractive rate of return i) --- the concept of opportunity cost is not discussed here due to the limitation of the length of the article). According to the criteria of the Incremental IRR Analysis, CFPs (3) and (4) are equally good. Next, choose either CFP (3) or (4) to compare with CFP (1). If CFP (3) is chosen, the (Incremental NCFs)$_{1\text{-}3}$ are (-6000; 2600; 2400; 2200) and its Incremental IRR is 10% per period. If CFP (4) is chosen, the (Incremental NCFs)$_{1\text{-}4}$ are (-9000; 3900; 3600; 3300) and its Incremental IRR is also 10% per period. All these mean that CFPs (1), (3) and (4) are equally good. Such a result is consistent with the NPV ranking that the three CFPs are equally good (i.e., having the same NPVs equal to 2,434.26).

Second, it is necessary to explain why IRR is not good to rank mutually exclusive multiple alternatives but good to be a financial indicator for a single project's financial strategy.

From the example of CFPs (1), (3) and (4) above, one can see that the IRR, unlike NPV, is "capricious" or "fickle". The IRR of an investment changes when the financial arrangement changes. Such changes do not happen in NPV and a rigorous mathematical proof on this was given in Tang and Tang (2003). Because NPV (at the minimum attractive rate of return) remains to be the same constant value for all the other associated CFPs derived from the original CFP, this constant (economic) value is much more reliable than the varying IRR values. Hence, the NPV was proposed to be called an economic indicator because it represents the economic value of an investment; it does not change but has only one constant value. The IRR, however, is situational and unstable, and it changes when the financial arrangement of an investment changes. Hence, it is not suitable to be used for ranking mutually exclusive multiple alternatives. Tables 1 and 3 reveal the unreliable result of the IRR ranking and the reliable result of the NPV ranking when mutually exclusive multiple alternatives are to be ranked. The IRR, however, is suitable to be a financial indicator for ranking the PFAs (project financing alternatives) of a single (not multiple) alternative.

4. IRR as a financial indicator

As said above, IRR is a financial indicator. Private investors usually like to play around with financial arrangements to optimize the rate of return under the circumstances of limited availability of funds. For example, CFP (1), that is (-10000; 5000; 5000; 5000), only gives an investor a constant rate of return of 23.4% per period for three periods. In other words, the investor will pay out 10,000 and obtain 15,000 (i.e., 5000 + 5000 + 5000; time value of money has not been considered). But if one divides the equity capital into ten equal shares (1,000 each share), then he or she can invest in ten CFP (4), that is (-1000; 1100; 1400; 1700), and obtain a constant rate of return of 113.1% per period for three periods. In other words, the investor will pay out 10,000 but obtain 42,000 (i.e. (1100 + 1400 + 1700) × 10); time value of money has not been considered). This is why IRR is a financial indicator. It is a financial indicator because it varies and tells the investors how to choose the optimum financial strategy under different financial constraints for a single investment project. Readers are recommended to read this current article in conjunction with Tang and Tang (2003) to fully comprehend that the NPV is an economic indicator and the IRR a financial indicator.

5. Misleading NPV-function profiles?

Hajdasinski (2004, pages 194-195) opined the NPV-function profile in Figure 1 of Tang and Tang (2003, page 70) is misleading and oversimplified. That Figure (called Figure 1 in this current article too) is reproduced below for easy reference.

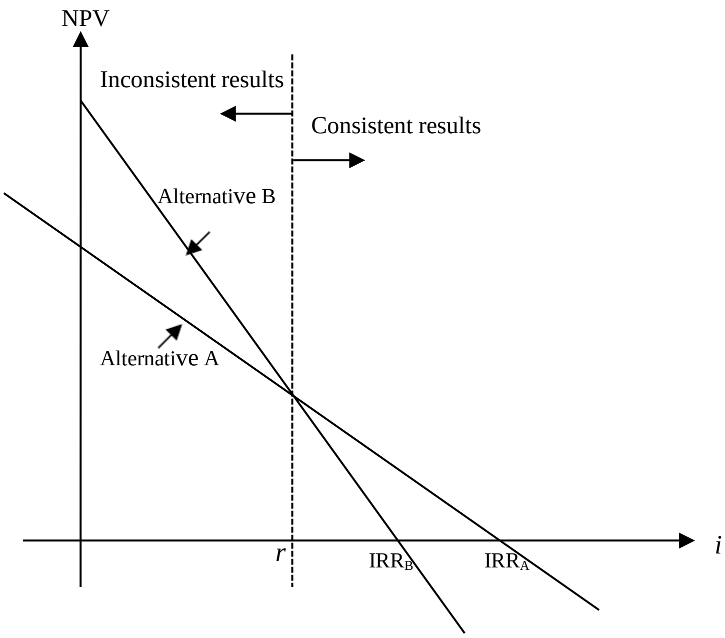

Fig. 1: Potential Inconsistency of NPV and IRR

Alternatives A and B in Table 1 is used to illustrate what this diagram tells, though the author agrees with Hajdasinski (2004) that the two straight lines representing Alternatives A and B are oversimplified. They are oversimplified firstly because they should not be straight lines but curves (close to straight line curves) and the left side of the line for Alternative A should be shown as an asymptote. Secondly, the two curves could take such an approximate shape (for a portion of the curve actually) only if each of the NPV-functions (or the CFPs) has only one sign variation and hence has only one real value of IRR, like Alternatives A and B that is being referred to. This point will be discussed in greater detail later in this article. Nevertheless, Hajdasinski (2004) also said "using free-style graphs for conceptual illustration purposes is a common, helpful, and accepted practice; however, such graphs should always reflect the basic features of the functions involved". Here, the two alternatives A and B (see Table 1) can be reflected by the approximate shapes of the two lines in Figure 1, which is a "free-style graph" that is useful for conceptual illustration. The following is an illustration.

Relate the following values to Figure 1:
On the left hand side of the vertical dotted line of Figure 1 (i.e. for $i < r$)
NPV of Alternative A (assume i = 6% p.a.) = 21.57
NPV of Alternative B (assume i = 6% p.a.) = 26.35
(Alternative B has a higher NPV than Alternative A)
IRR of Alternative A = 9.8% p.a.
IRR of Alternative B = 9.3% p.a.
(Alternative A has a higher IRR than Alternative B)
So, the NPV ranking is inconsistent with the IRR ranking.

Next, on the right hand side of the vertical dotted line of Figure 1 (i.e. for $i > r$)
NPV of Alternative A (assume i = 9% p.a.) = 3.42
NPV of Alternative B (assume i = 9% p.a.) = 1.92
(Alternative A has a higher NPV than Alternative B)
IRR of Alternative A = 9.8% p.a.
IRR of Alternative B = 9.3% p.a.
(Alternative A has a higher IRR than Alternative B)
So, the NPV ranking is consistent with the IRR ranking.

The value of r (as shown in Figure 1) is about 8.5 % p.a., which is between 6% p.a. and 9% p.a. Therefore, the above illustration explains what Figure 1 describes, and hence explains both Tables 1 and 3 for Alternatives A and B.

6. Multiple IRRs of a CFP

Hajdasinski (2004, page 193) also said Tang and Tang (2003) did not address the problem of the financial interpretation of multiple IRRs. The multiple IRR problem was not addressed in Tang and Tang's (2003) article because the main purpose of it was to propose that IRR is a financial indicator and NPV an economic indicator and this proposal was supported by a rigorous mathematical proof in that article. The multiple IRR issue seemed to be quite irrelevant to it. The author feels, however, now is a suitable occasion to address the multiple IRR problem.

Multiple IRRs occur when a CFP contains more than one sign variations, or in other words, when an NPV-function contains more than one sign changes. The author wishes to use the same CFP, here called CFP (6), that was originally used and called CFP (14) by Hajdasinski (2004, page 192). This CFP is reproduced below.

$$(-8000;\ 39200;\ -62500;\ 32200) \tag{6}$$

The NPV of CFP (6) shows a positive economic value/return of 175.81 at the minimum attractive rate of return i = 10%.p.a. (one period is assumed to be one year), indicating that it is a viable project because the NPV 175.81 is a positive value. The NPV result is always true, usable and reliable when based on a reasonable MARR (minimum attractive rate of return). For finding the IRR of CFP (6), there are three sign variations in CFP (6) so there are probably three positive real roots (IRRs). But in order to prove that this is absolutely true, CFP (6) has to be transformed into a polynomial equation (see (7) and (8) below).

$$\mathrm{NPV} = -8000 + \frac{39200}{(1+i)} - \frac{62500}{(1+i)^2} + \frac{32200}{(1+i)^3} \tag{7}$$

When the NPV in (7) is set equal to 0, it becomes the polynomial equation (8) as shown below, which can be used for exploring the number of real IRRs and evaluating them.

$$-8000\, i^{3} + 15200\, i^{2} - 8100\, i + 900 = 0 \qquad (8)$$

To explore the three roots (IRRs) of equation (8) to see whether they are real and positive, or real and negative, or complex-valued, one may apply the Descartes' Rule of Signs (Brodie, from Wikipedia) testing procedures as follows:

$f(i) = -8000\, i^{3} + 15200\, i^{2} - 8100\, i + 900$ --- 3 sign variations so there are 3 or 1 positive real root(s);
$f(-i) = 8000\, i^{3} + 15200\, i^{2} + 8100\, i + 900$ --- zero sign variation so there is no negative real root.
So, there are two possibilities: (1) 3 positive real IRRs, or (2) 1 positive real IRR and 2 complex IRRs.

As an important remark, readers' attention is drawn to the fact that the NPV versus *i* curve (Figure 1) for a CFP of more than one sign variations in *f(i)* will have entirely different shapes and will no longer looks like the shape that is shown in Figure 1. Only a CFP with one sign variation (like all examples prior to this section) will have an approximate shape as shown in Figure 1 and has only one real IRR.

Three positive real IRRs (without complex-valued IRR) are found, that is possibility (1) as said above, by solving the polynomial equation (8). The solutions are 15% p.a., 75% p.a. and 100% p.a. The same were found by Hajdasinski (2004) too. From appearance, none of the three IRRs is meaningful. Just by looking at the magnitude of the NCFs of CFP (6) and its NPV (+175.81 only), and by common sense, one cannot be convinced by any meaningful reason that the IRR could be 15% p.a., not to say 75% p.a. or 100% p.a. This will be further discussed a bit later.

Another method of applying Descartes' Rule of Signs is as follows:
Let $y = 1 + i$, then (7) becomes:

$$\text{NPV} = -8000 + \frac{39200}{y} - \frac{62500}{y^2} + \frac{32200}{y^3} \qquad (9)$$

When the NPV is set to be zero, (9) becomes:

$$-8000y^3 + 39200y^2 - 62500y + 32200 = 0 \qquad (10)$$

Apply the Descartes' Rule of Signs to (10):
$f(y) = -8000y^3 + 39200y^2 - 62500y + 32200$ --- 3 sign variations so there are 3 or 1 positive real root(s);
$f(-y) = 8000y^3 + 39200y^2 + 62500y + 32200$ --- zero sign variation so there is no negative real root.
So, there are two possibilities: (1) 3 positive real roots, or (2) 1 positive real root and 2 complex roots.

Solving equation (10), the roots are 1.15, 1.75 and 2.00, and this is possibility (1). By substituting these values into $y = 1 + i$, the IRRs (*i* values) are calculated to be 0.15, 0.75 and 1.00, which are similar to the solutions of equation (8). From this example, it can be seen that the number of sign variations of CFP (6) is the same as that of (9) or (10), and the number of sign variations of CFP (6) therefore is directly related to the number of real IRRs of CFP (6). This is generally true for CFPs with two or more sign variations.

It has been said above that the IRRs obtained from solving (8) or (10) seem to be not meaningful. However, the few decades' long opinion that multiple IRRs are meaningless was found incorrect by Hazen as reported in his article on the new perspective on multiple IRRs (Hazen 2003). According to Hazen, each and every IRR calculated from a CFP, whether real and positive, or real and negative, or complex-valued, is meaningful. In order to explain Hazen's finding, CFP (6) is transformed, based on each of the three IRRs, into three investment streams --- either pure investment stream(s) or pure borrowing stream(s). The transformation is performed as follows:

If a CFP is defined by (NCF_0; NCF_1; NCF_2; …; NCF_t; …; NCF_n), then there are n investment streams associated with this CFP and each investment stream (C_0; C_1; C_2; …; C_t; …; C_n) is obtained by the following relations. Readers may refer to Hazen (2003) and/or Tang (2003, Chapter 3).

$C_0 = -NCF_0$
$C_t = (1+k)C_{t-1} - NCF_t$ where k is an IRR of the n IRRs
$C_n = 0$

For CFP (6), or (NCF_0; NCF_1; NCF_2; NCF_3) = (-8000; 39200; -62500; 32200), n = 3 years and therefore investment streams 1, 2 and 3 are having *k* equal to 15% p.a., 75% p.a. and 100% p.a. respectively. Table 6 below shows how these investment streams are calculated.

Table 6: The three investment streams (C_0; C_1; C_2; C_3) associated with the (NCF_0; NCF_1; NCF_2; NCF_3) of CFP (6)

	C_0	C_1	C_2	C_3	NPV at *i* = 10% p.a.
Stream 1 (*k* =15% p.a.)	C_0 = $-NCF_0$ = 8000	$C_1 = (1+k)C_0 - NCF_1$ = (1+*k*) (8000) – 39200 = -30000	$C_2 = (1+k)C_1 - NCF_2$ = (1+*k*)(-30000) – (-62500) = 28000	$C_3 = (1+k)C_2 - NCF_3$ =(1+k)(28000) – 32200 = 0	+3867.77
Stream 2 (*k* = 75% p.a.)	C_0 = $-NCF_0$ = 8000	$C_1 = (1+k)C_0 - NCF_1$ = (1+*k*) (8000) – 39200 = -25200	$C_2 = (1+k)C_1 - NCF_2$ = (1+*k*)(-25200) – (-62500) = 18400	$C_3 = (1+k)C_2 - NCF_3$ =(1+k)(18400) – 32200 = 0	+297.52
Stream 3 (*k* =100% p.a.)	C_0 = $-NCF_0$ = 8000	$C_1 = (1+k)C_0 - NCF_1$ = (1+*k*) (8000) – 39200 = -23200	$C_2 = (1+k)C_1 - NCF_2$ = (1+*k*)(-23200) – (-62500) = 16100	$C_3 = (1+k)C_2 - NCF_3$ =(1+k)(16100) – 32200 = 0	+214.88

From Table 6, all streams 1, 2 and 3 are called pure investment streams because each of all the three (C_0; C_1; C_2; C_3) have a positive NPV at *i* = 10% p.a. (see the last column). Hypothetically speaking, if the NPV of (C_0; C_1; C_2; C_3) is negative at *i* = 10% p.a., the stream is called a pure borrowing stream. For CFP (6), there is no pure borrowing stream and all the three streams are pure investment streams. And according to Hazen's criteria, for stream 1, *k* = 15% p.a. > *i* = 10% p.a., CFP (6) is viable since it is a pure investment stream. Similarly, for stream 2, since *k* = 75% p.a. > *i* = 10% p.a., CFP (6) is viable because it is a pure investment stream. For stream 3, *k* = 100% p.a. > *i* = 10% p.a., hence CFP (6) is viable again for the same reason. Theoretically speaking, if there is an investment stream that is a pure borrowing stream, then *k* < *i* is needed in order to be a viable CFP, and vice versa. The results or the viability obtained from all the streams, according to Hazen (2003), are bound to be consistent with each other and are always consistent with the NPV evaluation of the CFP (i.e. the stream of NCFs) using *i*, and in this case, *i* = 10% p.a. This means that if one knows only one (and anyone) IRR of the multiple IRRs of a CFP, he or she will be able to evaluate the viability of the CFP with that known IRR using the method just described.

The multiple IRRs, therefore, do not cause any "so-far-unresolved financial interpretation problems", a question raised by Hajdasinski (2004, page 193). Hazen's method is, besides positive IRRs, also applicable to negative IRRs and complex IRRs (Hazen 2003) and examples are given in Hazen's article. Therefore, for the Incremental IRR Analysis the current author describes in the early part of this article, using only one (and anyone) Incremental IRR obtained from a stream of Incremental NCFs (e.g., CFP_B - CFP_A) is sufficient to compare two CFPs at a time and this should always be successful. The Incremental IRR Analysis is always workable and the multiple IRR issue is not a hindrance to this Analysis. If the Incremental IRR is a single and real IRR, the use of only the criteria stipulated before Table 2 in Section 2 is already sufficient. But if the Incremental IRR is one of the multiple Incremental IRRs, then the criteria described following Table 6 will prevail. Of course, the latter criteria are applicable also to the single and real Incremental IRR case by substituting *k* equal to this single and real Incremental IRR.

7. The multiple real IRRs calculated from a CFP are not the CFP's true rates of return

At this juncture, it must be pointed out that the IRR can truly represent a CFP's rate of return only if the line (or more accurately, curve) representing the investment is of the approximate shape as shown in Figure 1 such that the CFP has only one real IRR. A related remark has already been given in the paragraph immediately before Table 2 in Section 2. Furthermore, the example explaining why IRR is called a financial indicator in the paragraph under the Section "IRR as a financial indicator" is true only for a CFP with only one real IRR. In these cases, the magnitude of the single real IRR obtained from a CFP can truly represent the rate of return of that CFP. However, for a CFP with multiple IRRs, the magnitudes of the IRRs do not represent the true rates of return. The multiple IRRs in such cases are useful only for transforming a CFP into a number of (C_0; C_1; C_2; …; C_n), the pure investment streams or the pure borrowing streams, by which the investment decision can be made. But for these multiple IRRs, Hazen (2003, page 46) said "the magnitude of the IRR by itself carries no further information". For example, 15% p.a., 75% p.a., or 100% p.a. are the multiple IRRs of CFP (6); none of them represents the true rate of return of CPF (6) --- (-8000; 39200; -62500; 32200).

Similarly, the CFP (-4; 12; -9) given as example by Magni (2010, page 151) has two double roots (for y = 1+ *i*) as follows:

$$-4y^2 + 12y - 9 = 0$$
$$\text{or } (2y - 3)\,(2y - 3) = 0$$

So, y = 1.5 and 1.5 (double roots), but since y = 1+ *i*, the IRRs (i.e., *i*) are 0.5 (50%) and 0.5 (50%). By common sense, the CFP (-4; 12; -9) can in no way have a rate of return of 50%. The NPV is -0.53, which is reasonable for the CFP (-4; 12; -9). In Robison *et al*. (2015, page 500), Magni's (2010) CFP (-4; 12; -9) was changed to (-4; 12; -8). This example has been mentioned earlier at the end of Section 2. Robison *et al*. calculated the IRR of it to be 0% and the NPV to be 0.30. The IRR decreases from 50% to 0% while the NPV increases from -0.53 to +0.30 by assuming a 10% discount rate. The IRRs calculated are entirely lack of sense but the NPVs are sensible. The magnitudes of the multiple IRRs are not representing the true rates of return and this confirms Hazen's statement "the magnitude of the IRR by itself carries no further information" for multiple IRR cases (Hazen 2003, page 46).

8. Modified IRR and Modified NPV (Robison *et al*. 2015)

The idea of working out a MIRR (modified IRR) and a MNPV (modified NPV) is also proposed to attempt to find a consistent IRR and NPV rankings. The title of the paper "Consistent IRR and NPV rankings" (Robison *et al*. 2015; Lefley 2018) is somewhat misleading because actually it mainly talks about consistent MIRR and MNPV rankings but not consistent IRR and NPV rankings. The concept of calculating MIRR and MNPV is somehow similar to that of calculating ERR (external rate of return). For this concept, the number of sign variations of a CFP is reduced to one by transforming the CFP by means of periodic reinvestments based on the minimum attractive rate of return so only one real IRR will occur at the end (Tang 2003, Chapter 3, pages 47-51)). MIRR and MNPV, in the current author's opinion, are only supplementary methods to the IRR/NPV. There are also two major drawbacks. The following is a brief discussion of the article in question.

For alternatives with no initial investment size differences, they should not be called, as discussed earlier, mutually exclusive investment alternatives but homogeneous investment size alternatives. As given as an example in Table 4 of this current article, the IRR ranking must be consistent with the NPV ranking for homogeneous investment size alternatives. According to Robison *et al*. (2015), the mutually exclusive investment alternatives could have their initial investment size differences adjusted and made equal by scaling or adding approaches. The former approach leads to evaluating scaled MIRR/MNPV and the latter approach to evaluating added MIRR/MNPV, with both approaches using a periodic reinvestment calculation. From Table 4 of this current article, since the initial investment sizes are made equal, the scaled MIRR ranking must be consistent with the scaled MNPV ranking. Similarly, the added MIRR ranking must be consistent with the added MNPV ranking. However, the scaled MIRR/MNPV rankingmay not be consistent with the added MIRR/MNPV ranking. This is not so appealing and is a drawback. Furthermore,the calculation as shown in Table III on page 507 of Robison *et al*. (2015) would not be possible if the investment alternatives have multiple IRRs. It is also mentioned in Table V on page 512 of that article that no ranking is possible if multiple IRRs exist. This is another drawback. But for the Incremental IRR Analysis described earlier, it is immune to multiple IRR problems. That is why the current author considers that the Incremental IRR Analysis is more appealing, although it is not the main theme of this article. After all, the IRR and the NPV rankings, in the current author's opinion, need not be consistent provided the concepts on all the issues are clearly understood.

9. Misreading of IRR's real proposition?

Silva *et al*. (2018, page 160) wrote "Due to potential flaws of the IRR, academics have a preference for the NPV method. This was also said by Osborne (2010) eight years earlier than Silva *et al*. However, practitioners apply IRR more frequently than NPV to analyze their investments (Magni 2014). As a remark of the current author to the above statement, the NPV is a number (or figure) that is quite cognitively inefficient, unlike the IRR that can give investors a definite rate of return in percentage per period. The latter is easier than the former to be understood by an investor what he or she can get from an investment, and so probably because of this, practitioners (usually less academic) like to adopt the IRR approach. On the other hand, academics have a preference for the NPV approach because its concept is sound although it (the number or figure) is a bit harder to comprehend. This reason was also observed by Lefley (2018, page 48). Lefley, moreover, mentioned that MGR (marginal growth rate) (Lefley 2015) is a natural extension of NPV, and the MGR (expressed in the form of a rate) should act in the latter's support. Whenever a project shows a positive NPV it will also have an MGR greater than zero and therefore support the same accept/reject decision for mutually exclusive projects (Lefley 2018, page 50). The current author, however, considers that the MGR is, as also said by Lefley, supplementary to the NPV, and because of the need to keep this article within its purpose, MGR is not going to be discussed further here.

Furthermore, Silva *et al*. (2018, page 160) continued to say "there are researchers who assert that the NPV versus the IRR debate only exists because of a misreading of IRR's real proposition", and then the article of Tang and Tang (2003) was cited at that point by Silva *et al*. Tang and Tang (2003) mentioned that it was pointed out in an article related to linear programming (Battaglio *et al*. 1996; Tang 1999) that IRR is meant for a consumer's point of view and NPV for a banker's point of view. To this point, Tang and Tang (2003) added that Battaglio *et al*.'s definitionis quite close to the true definition as the consumers usually have relatively limited money and the banks usually have relatively unlimited money. Tang and Tang (2003) then further added that an even more fundamental definition can be given, that is, the IRR gives the private investor's point of view and the NPV the society's (or the community's) point of view. In other words, the IRR is a financial indicator and the NPV an economic indicator. Therefore, togetherwith the earlier arguments in several different Sections in the current article, there should not be a misreading of IRR'sreal proposition, as Silva *et al*. questioned. The IRR is a financial indicator and is different from the economic indicator NPV intrinsically. This has been explained by a number of examples in this current article. Finally, the author wants to share his experience with the readers: the NPV method is always simple, correct, and accurate; it avoids all the troubles associated with the IRR method. The NPV is the best method for engineering economists.

10. Conclusions

This article firstly demonstrates that the NPV ranking of mutually exclusive multiple alternatives is always correct if the MARR (minimum attractive rate of return) is based. The IRR ranking is not always correct. The IRR ranking is unreliable and is therefore not recommended. The Incremental IRR Analysis (not direct IRR ranking), however, is always consistent with the NPV ranking, and hence is always correct in ranking mutually exclusive multiple alternatives.

When the Incremental IRR Analysis faces the multiple Incremental IRR problem, no matter the Incremental IRRs are real and positive, or real and negative, or complex-valued, the same procedure can be applied in performing the Analysis by using anyone of these multiple Incremental IRRs to rank two alternatives at a time. The procedure is to find an investment stream (C_0; C_1; C_2; …; C_n) by the use of anyone Incremental IRR (or k) of the multiple Incremental IRRs. Then test whether it is a pure investment stream or a pure borrowing stream, and by comparing k with i (MARR), the two alternatives can be ranked (Hazen 2003). Then, two by two at a time and step by step, all alternatives can be ranked.

Then, the function of NPV as an economic indicator and that of IRR as a financial indicator is illustrated by an example (Tang and Tang 2003). The economic indicator NPV should be used to rank multiple projects, but the financial indicator IRR should be used to rank financial arrangements of a single project. The term IRR ranking, however, is used widely to mean ranking multiple projects by ranking their direct IRRs. Hence, that a financial indicator which is only suitable to rank financial arrangements of a single project is wrongly used to rank multiple projects (in an economic sense) is bound to cause problems.

Descartes' Rule of Signs is then applied to explain the total number of real and positive, real and negative, and complex-valued IRRs in a CFP (cash flow pattern). Examples taken from Hajdasinski (2004), Magni (2010), and Robison et al. (2015) are used to illustrate Descartes' Rule and multiple IRRs. Each of the multiple IRRs, no matter real and positive, real and negative, or complex-valued, is meaningful and can be used to make investment decisions (Hazen 2003). Each and anyone of them gives the same result of the investment decision.

In the author's opinion, all other methods such as the MIRR (modified internal rate of return), the MGR (marginal growth rate), the Incremental IRR Analysis, etc. are only supplementary to the NPV method. The NPV method is simple and easy to use for finding the economic return (or economic value) of an investment or for ranking investment alternatives. Throughout the article, the author uses the Incremental IRR Analysis, the comparison of three CFPs with the same NPVs but different IRRs, the multiple IRR theory, and the IRR's real proposition to illustrate that the NPV is an economic indicator and the IRR a financial indicator. The financial indicator IRR is incorrect to be used to rank multiple mutually exclusive investment alternatives, but is useful to be applied by investors to find out the best financial strategy or arrangement to obtain an optimal gain from a single investment project. The economic indicator NPV can tell investors their gain or loss accurately and is suitable for ranking multiple mutually exclusive investment alternatives. The NPV is always simple, correct, and accurate. It is the best method for engineering economists.

11. Note

(1) In the previous article (Tang and Tang 2003), there is a printing error in the last line of page 70: "as the consumersand the banks usually have relatively unlimited money" should read "as the consumers usually have relatively limited money and the banks usually have relatively unlimited money". This change has been announced several quarters later in an issue of *The Engineering Economist*. (2) Another printing error that has not been announced is the last item(Item 12) in the References of the same article that "2nd edition" should read "3rd edition".

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