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Longest Path Time-Cost Analysis of Construction Projects with Generalised Activity Constraints

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Abstract
Time-cost analysis is an important element of project scheduling, especially for lengthy and costly construction projects, as it evaluates alternative schedules and establishes an optimum one considering any project completion deadline. Existing methods for time-cost analysis have not adequately considered typical activity and project characteristics, such as generalised precedence relationships between activities and external time constraints, that would provide a more realistic representation of actual construction projects. The present work aims to incorporate such characteristics in the analysis and proposes a method for developing optimal project time-cost curves based on critical path analysis. In this method, the project is described through a matrix where all paths are tabulated with respect to activities. The project matrix includes values of 1, 0, or -1 depending on the type of precedence relation between an activity and its adjacent ones within a path. Integer programming is employed to choose among all activity crashing alternatives those which reduce path durations to a desired project length value with the lowest possible crashing cost. Evaluation results indicate that the method can be reliably applied to construction projects with the above characteristics.

Keywords
Time-Cost Trade-Off, Project Planning, Integer Programming, Optimisation, Project cost

1. Introduction

The time-cost trade-off problem has extensively been studied for more than four decades and has been recognised as a particularly difficult combinatorial problem. Existing methods for time-cost trade-off analysis have dealt with a simplified project representation, e.g., considering sequential execution of project activities ("finish-to-start" dependency between activities). It is known, however, that in construction projects, activities are often scheduled concurrently in order to shorten the project duration. For instance, construction of the sub-base, base, and asphalt layers in a road section, although logically sequential activities, can be scheduled partly contemporaneously allowing an appropriate time lag between their initiation. Such cases are modelled in scheduling analysis with generalised precedence relations, such as "start-to-start", and "finish-to-finish" including lead or lag time. Further, external time constraints (e.g., "start no earlier than" or "finish no later than") may be imposed to certain activities, sub-projects, or the project following necessary technical restrictions or managerial/political decisions. For instance, an activity that requires a special type of machinery may not start until the machinery is available on-site. Such project characteristics and constraints are important for a realistic simulation of construction projects.

The literature on the time-cost trade-off problem is rich and this indicates the scientific interest on this subject. Proposed solution methods have employed linear, integer, or dynamic programming, other (heuristic) methods and, lately, genetic algorithms. The methods that appear in the literature can be classified into the following categories. The first includes exact methods based on linear and/or integer programming to solve the basic time-cost trade-off problem (e.g., Liu et al, 1995). Approximate methods, in the second group, rely on decomposition approaches (Panagiotakopoulos, 1977, De et al, 1995, Chassiakos et al, 2000) or genetic algorithms (Li et al, 1999, Zheng et al, 2004) with a major objective to reduce the computational effort that is required by methods of the previous group. Finally, a few methods have gone beyond the basic problem and attempted to attack more realistic project cases considering, for instance, generalised time relations among project activities (Elmaghraby and Kamburowski, 1992, Bartusch et al, 1988, Sakellaropoulos and Chassiakos, 2004), or the uncertainty associated with the problem parameters (Feng et al, 2000, Isidore and Back, 2001). Limitations of existing methods include inadequate modelling of real-life project characteristics (e.g., project structure, activity dependence), inclusion of quite restrictive assumptions (e.g., activity time-cost functions), or computational inefficiency.

This paper presents a method for developing optimal time-cost curves in the general case of a project with generalised precedence relations among activities and external time constraints imposed to certain activities or parts of the project. According to this method, the project is initially represented in terms of its paths and activities. Following, integer programming is employed to determine the best combination of activity durations which results in a desired project length. The proposed formulation is elementary and intuitive and allows easy application. In addition, it often results in smaller problem size compared to that of previous methods.

2. The Proposed Method

The project cost depends (among other factors) on the required (or desired) time to complete the project. Considering the project structure (its activities and the sequence of operations) and that an activity can generally be completed in a number of alternative ways (associated with particular duration and cost values), the objective is to select the appropriate execution option so that the project is completed by a desired deadline and in an optimal way, i.e., with the minimum cost. If the analysis is repeated for any feasible project length, an optimal time-cost curve is developed for the project which can also be used for optimising project duration. To better simulate construction projects, generalised precedence relations between activities, and external time constraints are modelled in this work. In particular, precedence relations of the form "Finish-to-Start" (fs), "Start-to-Start" (ss), and "Finish-to-Finish" (ff) are considered. Lead or lag times may accompany these relations. External time constraints refer to cases such as start/finish no earlier than/no later than/on time *D*.

The proposed method is based on the development of the so-called project matrix. Each row of this matrix refers to a project path while each column represents a project activity. The *(i,j)*-th element of the project matrix, denoted by *p(i,j)*, takes a value of 0 if activity *j* does not belong to path *i*, and values according to Table 1 if activity *j* belongs to path *i*. The value *p(i,j)* in the latter case depends on the type of relations between activity *j* and its adjacent activities *j-1* and *j+1* in path *i*.

Table 1: Values for Project Matrix according to Precedence Relations

Activity *j-1*									
Precedence relation	fs	fs	fs	ss	ss	ss	ff	ff	ff
Activity *j*	1	0	1	1	0	1	0	-1	0
Precedence relation	fs	ss	ff	fs	ss	ff	fs	ss	ff

Activity *j+1*	

The duration of each path is calculated as the sum of durations of the activities included in the path, plus any lag/lead time contribution, i.e., the duration *D(i)* of path *i* is given by

$$D(i) = \sum_{j=1}^{n} p(i,j)\, d(j) + \sum_{j=1}^{n} l(i,j), \quad i = 1, 2, \ldots, n_p, \tag{1}$$

where *d(j)* is the duration of activity *j*, *p(i,j)* is the *(i,j)*-th element of the project matrix, *l(i,j)* is the lag or lead time (with minus sign) of activity *j* associated with path *i*, *n* is the number of project activities, and n_p is the number of paths. The project matrix is sorted in descending order of path length and is augmented with information regarding activity durations and crashing costs. The longest path (or paths) of the network represents the critical path and its length determines the project duration.

The second part of the method constitutes the optimisation process. Starting from the normal project length, the objective is to expedite the project in an optimal way. The formulation requires examining all project paths with longer duration than the desired project length. A zero-one variable *y(j,k)* is defined for each activity to indicate whether activity *j* will be crashed by *k* time steps in order to achieve the desirable project length. The integer program is written as:

$$\text{minimize} \sum_{j=1}^{n} \sum_{k=1}^{K(j)} c(j,k)\, y(j,k) \tag{2}$$

subject to

$$\sum_{j=1}^{n} \sum_{k=1}^{K(j)} p(i,j)\, \Delta d(j,k)\, y(j,k) \geq D(i) - D_d, \quad i = 1, 2, \ldots, n_{cp}, \tag{3}$$

$$\sum_{k=1}^{K(j)} y(j,k) \leq 1, \quad j = 1, 2, \ldots, n, \quad y(j,k) = 0 \text{ or } 1, \tag{4}$$

where *K(j)* is the number of crashing steps for activity *j*, *c(j,k)* is the cost for crashing activity *j* for the first *k* steps, *Δd(j,k)* is the duration reduction if activity *j* is crashed by the first *k* steps, *D(i)* is the duration of path *i*, D_d is the desired (target) project duration, and n_{cp} is the number of "critical" paths, i.e., those with $D(i) > D_d$. The objective function (2) represents the total crashing cost. The set of constraints (3) ensure that each path is adequately reduced in length to conform to the desired project duration. Finally, the set of constraints (4) prevent activity crashing phases from being double-counted. To find the optimal crashing strategy at various project durations, the IP model attains the same form except that additional constraints (3) may be required according to "critical" paths.

The incorporation of external constraints is done with a similar process. In particular, for an external constraint of the form "start (finish) activity *m* not later (earlier) than D_m", a subproject is considered which consists of all activities that precede activity *m*. In the IP model, additional constraints, similar to the ones shown in (3), are employed for the subproject to describe the external constraint. On the other hand, external constraints of the form "start (finish) activity *m* not earlier (later) than D_m" are treated similarly to the previous case, with the exception that the subproject now consists of all activities following and affected by activity *m* up to the project end.

In terms of the expected computational efficiency of the model, it is known that the computational effort associated with an exact solution approach would grow exponentially with the problem size (De et al, 1995).

The proposed model cannot escape of such rule. Assessing the size of the proposed model indicates that the number of required zero-one variables equals the number of possible crashing steps of all activities. The number of constraints is at most equal to the number of paths (for the project and any subproject) plus the number of activities that have more than one crashing step. The proposed method is more efficient than previous methods in certain cases. This improvement mainly appears in small or moderate size project networks but efficiency is reduced in larger networks. The limitation in the latter case results from the need to record all network paths.

3. An Application Example

An example has been structured and presented to illustrate the model application. The precedence relations between project activities and the alternative time-cost options for each activity are presented in Table 2 while the corresponding network diagram is shown in Figure 1. In addition, an external constraint has been considered requiring that activity G cannot start earlier than 14 weeks after the project commencement. The project matrix is shown in Table 3. Paths have been sorted in decreasing order of their length. The normal project duration is 30 weeks as determined by the critical path A-C-E-F-H-I.

Table 2: Activity Precedence Relations and Time-Cost Options

Activity	No	**Precedence Relations**	Option 1		Option 2		Option 3	
			Time	***Cost***	Time	***Cost***	Time	***Cost***
A	1	-	6	18	5	21	4	25
B	2	A(ss+2)	7	15	6	17	5	20
C	3	A(fs-1)	10	25	8	30	6	35
D	4	A(fs+3)	6	12	5	16	-	-
E	5	B(ff+3), C(ss+2)	6	16	5	20	-	-
F	6	B(fs), E(ss+5)	9	22	7	26	6	30
G	7	C(ff+2), D(fs)	8	21	7	24	6	30
H	8	E(fs), F(fs)	5	15	4	18	3	22
I	9	G(ff+3), H(fs)	4	12	3	14	-	-

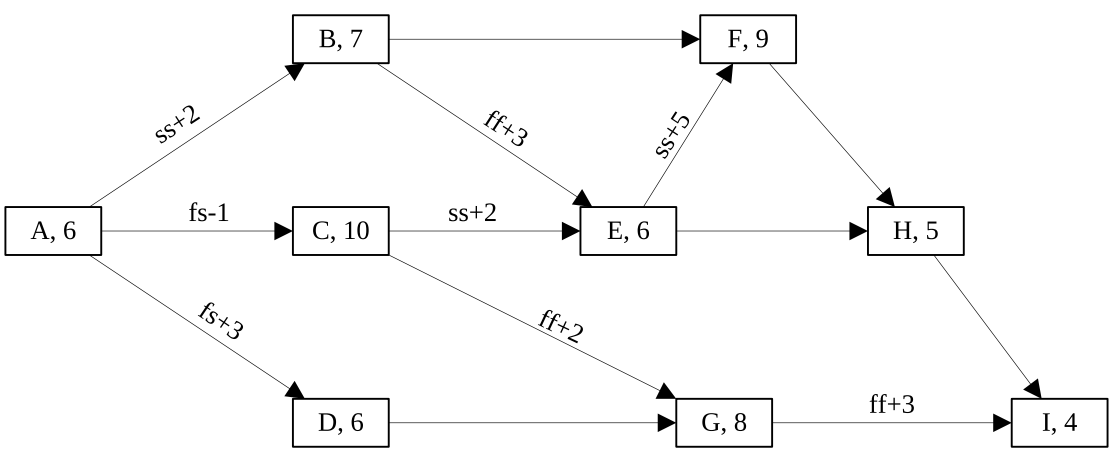

Figure 1: AON Network for the Example Project

Table 3: Project Matrix and Path Lengths for Normal Activity Execution

	Activity j	A	B	C	D	E	F	G	H	I	Σl_i	Path length
	Normal duration	6	7	10	6	6	9	8	5	4		
	Crash duration 1	5	6	8	5	5	7	7	4	3		
	Crash duration 2	4	5	6			6	6	3			
	Crash cost 1*	3	2	5	4	4	4	3	3	2		
	Crash cost 1&2*	7	5	10			8	9	7			
	A-C-E-F-H-I	1					1		1	1	6	30
P	A-B-E-F-H-I		1			-1	1		1	1	10	29
A	A-B-F-H-I		1				1		1	1	2	27
T	A-D-G-I	1			1			1			6	26
H	A-C-E-H-I	1				1			1	1	1	22
S	A-B-E-H-I		1						1	1	5	21
	A-C-G-I	1		1							4	20
	Constraint (G-I)							1			3+14	25

* The crash cost 1 represents the cost difference between Option 1 and Option 2 of Table 1 while the crash cost 1&2 the cost difference between Option 1 and Option 3.

The integer program is presented here for the indicative case of 24-week target duration. In the following relationships, y_{A1} refers to the first crashing step of activity A while y_{A12} to both steps together (similar notation applies to other activities). The problem is written as

$$\text{minimize} \quad 3y_{A1} + 7y_{A12} + 2y_{B1} + 5y_{B12} + 5y_{C1} + 10y_{C12} + 4y_{D1} + 4y_{E1} + 4y_{F1} + 8y_{F12} + 3y_{G1} + 9y_{G12} + 3y_{H1} + 7y_{H12} + 2y_{I1} \tag{5}$$

subject to

$$y_{A1} + 2y_{A12} + 2y_{F1} + 3y_{F12} + y_{H1} + 2y_{H12} + y_{I1} \geq 6 \tag{6}$$

$$y_{B1} + 2y_{B12} + y_{E1} + 2y_{F1} + 3y_{F12} + y_{H1} + 2y_{H12} + y_{I1} \geq 5 \tag{7}$$

$$y_{B1} + 2y_{B12} + 2y_{F1} + 3y_{F12} + y_{H1} + 2y_{H12} + y_{I1} \geq 3 \tag{8}$$

$$y_{A1} + 2y_{A12} + y_{D1} + y_{G1} + 2y_{G12} \geq 2 \tag{9}$$

$$y_{G1} + 2y_{G12} \geq 1 \text{ (external constraint)} \tag{10}$$

$$y_{A1} + y_{A12} \leq 1 \tag{11}$$

$$y_{B1} + y_{B12} \leq 1 \tag{12}$$

$$y_{C1} + y_{C12} \leq 1 \tag{13}$$

$$y_{F1} + y_{F12} \leq 1 \tag{14}$$

$$y_{G1} + y_{G12} \leq 1 \tag{15}$$

$$y_{H1} + y_{H12} \leq 1 \tag{16}$$

$$y_{A1}, y_{A12}, y_{B1}, y_{B12}, y_{C1}, y_{C12}, y_{D1}, y_{E1}, y_{F1}, y_{F12}, y_{G1}, y_{G12}, y_{H1}, y_{H12}, y_{I1} = 0, 1. \tag{17}$$

The optimal solution to this problem is given by

$$y_{A1} = y_{F1} = y_{G1} = y_{H12} = y_{I1} = 1,\ y_{A12} = y_{B1} = y_{B12} = y_{C1} = y_{C12} = y_{D1} = y_{E1} = y_{F12} = y_{G12} = y_{H1} = 0$$

at a total cost of 19.0 units. Table 4 summarises the output of the model for the whole range of feasible project durations.

To obtain an efficiency indication in terms of resource requirements, the size of the integer program is assessed and compared to the corresponding LP/IP program which performs a time-cost analysis based on activity start and finish times (Sakellaropoulos and Chassiakos, 2004). The proposed path-based model presents a varying problem size depending on the crashing stage. Instead, the structure of the activity-based model remains unchanged in size at any crashing phase. The assessment results in Table 5 (which refer to project target duration of 24 weeks) are in favour of the proposed model, however, this outcome cannot be generalised for any network type and size. This is because it is difficult to establish a robust methodology to compare requirements among different formulations and, thus, comparisons can be only made with regard to particular applications.

Table 4: Optimal Crashing Strategy

	Activities									
Project length	A	B	C	D	E	F	G	H	I	Crashing cost
30										0
29									1	2
28						1				4
27						1			1	6
26						1		1	1	9
25	1					1		1	1	12
24	1					1	1	2	1	19
23	1					2	2	2	1	29

Table 5: Problem size assessment

Model	Present method	Sakellaropoulos and Chassiakos, 2004
Number of non-integer variables	-	20
Number of binary variables	15	24
Number of constraints	11	69

4. Conclusions

Time-cost analysis is an important element of project scheduling, especially for lengthy and costly construction projects, as it evaluates alternative schedules and establishes an optimum one considering existing deadlines for project completion. In order to develop realistic time-cost curves for such projects, typical activity and project characteristics, such as generalised precedence relationships and external time constraints, should be modelled. The present work aims to incorporate such characteristics and has developed a method for obtaining optimal project time-cost curves based on critical path analysis. In this method, the project is described through a matrix where all paths are tabulated with respect to activities. The project matrix includes values of 1, 0, or -1 depending on the type of precedence relation between an activity and its adjacent ones within a path. Following, the path durations are calculated. Integer programming is employed to choose among all activity crashing alternatives those which reduce path durations to a desired project length value with the lowest possible crashing cost. The proposed formulation is intuitive and allows easy application. Further, it often results in smaller problem size compared to that of previous methods. Evaluation results indicate that the method can be reliably applied to construction projects with the above characteristics.

5. References

Bartusch, M., Möhring, R.H., and Radermacher, F.J. (1988). "Scheduling project networks with resource constraints and time windows". *Annals of Operations Research*, Vol. 16, pp. 201-240.

Chassiakos, A.P., Samaras, C., and Theodorakopoulos, D. (2000). "An Integer Programming Method for CPM Time-Cost Analysis". *Computer Modeling in Engineering and Sciences*, Vol.1, No 4, pp.9-18.

De, P., Dunne, E.J., Ghosh, J.B., and Wells, C.E. (1995). "The discrete time-cost trade-off problem revisited". *European Journal of Operational Research*, Vol. 81, pp. 225-238.

Elmaghraby, S.E., and Kamburowski, J. (1992). "The analysis of activity networks under generalized precedence relations". *Management Science*, Vol. 38, pp. 1245-1263.

Feng, C.W., Liu, L., and Burns, S.A. (2000). "Stochastic construction time-cost trade-off analysis". *Journal of Computing in Civil Engineering*, Vol. 14, No. 2, pp. 117-126.

Isidore, L.J., and Back, W.E. (2001). "Probabilistic optimal-cost scheduling". *Journal of Construction Engineering and Management*, Vol. 127, No. 6, pp. 431-437.

Li, H., Cao, J.N., and Love, P.E.D. (1999). "Using machine learning and genetic algorithms to solve time-cost trade-off problems (MLGAS)". *Journal of Construction Engineering and Management*, Vol. 125, No. 5, pp. 347-353.

Liu, L., Burns, S.A., and Feng, C.W. (1995). "Construction time-cost trade-off analysis using LP/IP hybrid method". *Journal of Construction Engineering and Management*, Vol. 121, No. 4, pp.446-454.

Panagiotakopoulos, D. (1977). "Cost-time model for large CPM project networks". *Journal of the Construction Division*, Vol. 103, pp. 201-211.

Sakellaropoulos, S., and Chassiakos, A.P. (2004). "Project time-cost analysis under generalised precedence relations". *Advances in Engineering Software*, Vol. 35, No. 10-11, pp. 715-724.

Zheng, D.X.M., Ng, S.T., and Kumaraswamy, M.M. (2004). "Applying a Genetic Algorithm-Based Multiobjective Approach for Time-Cost Optimization". *Journal of Construction Engineering and Management*, Vol. 130, No. 2, pp. 168-176.